Cournot-Nash equilibria and optimal transport

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1 Outline

This paper is intended as companion reading for Adiren Blanchet and Guillaume Carlier's paper Optimal Transport and Cournot-Nash Equilibira [BC16]. We assume a reasonable backgrund in the theory of OT, but no background in game theory or economics. We provide a thorough background so that Blanchet and Carlier's paper should be readily accessible upon reading our report. This project is completed as part of the course requirements for Optimal Transport + Economics, offered to students at universities forming the Pacific Institute for the Mathematical Sciences, and taught by Professor Brendan Pass of the University of Alberta.

In section 2, we begin with the bare essentials of game theory. We provide basic definitions and intuitive motivations for concepts, and conclude with Nash's theorem on the existence of equilibria in finite games. Section 3 extends the background on game theory to account for the kinds of games considered in [BC16]. We consider players and actions whose traits vary continuously, naturally introducing the need for measures. Section 2 concludes with results analogous to Nash's theorem, originally proved in [MC84].

Section 4 combines the basic game theory notions presented in sections 2 and 3 with the theory of optimal transport, as presented by Blanchet and Carlier. We explore the natural links between the two fields of studies, and prove equivalence between equilibria and optimal transport plans. We conclude by linking a well known charactarization of solutions to a wide class of OT problems wtih purity of strategies in game theory.

Finally, in section 5, we perform a high level investigation of some of the extensions proposed by Blanchet and Carlier. We make no claim to fully cover every detail in the original paper, but rather provide a high level overview of some of the areas of exploration put forward in [BC16]. We discuss rivalry of use/strategy exhaustion, how it can be accounted for in the language of measure theory, and why its introduction fundamentally changes the analysis required to find equilbira in games. Section 6 concludes.

2 Game theory essentials

We now give a simplified overview of game theory. We define games, payoff functions, pure/mixed strategies, and Cournot-Nash equilibria. We conclude with Nash's theorem on the existence of Cournot-Nash equilibria.

Definition 2.1. Let $X = \{1, 2, \ldots, n\}$ and Y be finite sets. A function $\Phi : X \times Y^n \to \mathbb{R}$ is called a *payoff function*. Together, the triple (X, Y, Φ) is called a *finite game*.

Remark. The *players* in a game are indexed by X, and their available *actions* are the elements of Y. The payoff function Φ gives the reward $\Phi(x, (y_1, \ldots, y_n))$ to player $x \in X$ given each player $i \in X$ takes action y_i.

Example. Let $X = Y = \{1, 2\}$. We have a two player game, where each player has two possible actions. The payout function Φ can be most easily represented by "game matrices" A_x. For concreteness,

$$A_1 = \begin{pmatrix} 3 & 6 \\ 2 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 4 & 3 \\ 5 & 2 \end{pmatrix}$$

encode that $\Phi(1, ([1, 2], [2, 1])) = 2$ and $\Phi(2, ([1, 2], [2, 1])) = 5$. That is, the payouts to players 1 and 2 given that they play actions 2 and 1 are 2 and 5, respectively.

Definition 2.2. A *strategy* is a probability distribution on the set of actions, $\sigma \in \mathbb{P}(Y)$. A strategy is called *pure* when it assigns all mass to a single $y \in Y$. Otherwise, a strategy is called *mixed*. A matching of players to strategies is a function $\sigma(\cdot) : X \to \mathbb{P}(Y)$.

Remark. The domain of Φ is enlarged to $X \times \mathbb{P}(Y)^n$. The payout to player x_0 given all players' strategies σ_i is the expected payout to x_0. Symbolically, $\Phi(x_0, (\sigma_1, \ldots, \sigma_n)) = \mathbb{E}\left[\Phi(x_0, \text{ strat\_RVs})\right]$ where "strat_RVs" is a term representing an n-tuple of independent random variables S_i with respective laws σ_i for $1 \le i \le n$.

Definition 2.3. A collection of strategies $\sigma_1, \ldots, \sigma_n$ is called a *Cournot–Nash equilibrium* when no player is strictly better of by unilaterally changing their strategy. That is,

$$\sigma_i \in \operatorname{argmax}_{\tau \in \mathbb{P}(Y)} \Phi(i, (\sigma_1, \ldots, \sigma_{i-1}, \tau, \sigma_{i+1}, \ldots, \sigma_n))$$

for each i.

Example. In the example above, each player always choosing strategy 1 is a Cournot-Nash equilibrium which employs pure strategies.

Theorem 2.4. *Every finite game has at least one Cournot-Nash equilibrium.*

3 Games and measures

We generalize from n players to a distribution on player types. We define payoff functions, games, and Cournot-Nash equilibria in this more general context.

Definition 3.1. Let X and Y be a compact subsets of Euclidean space.[1] A pair (X, μ) where $\mu \in \mathbb{P}(X)$ is called a player space. A function $\Phi : X \times Y \times \mathbb{P}(Y) \to \mathbb{R}$ is a payoff function. A triple $((X, \mu), Y, \Phi)$ defines a game.

Remark. X and Y index player types and actions. $\Phi(x, y, \nu)$ gives the payoff to player x taking action y, given that $\nu \in \mathbb{P}(Y)$ is the relative abundance of different actions taken. Since the payoff function Φ depends on the aggregate of actions (and not on who plays which action), we call such games *anonymous*.

Remark (Many players). In some sense, we have a generalization of the finite game framework developed above. One might think of the set of n players $X = \{1, 2, \ldots, n\}$ as being the player space $(X, \sum_{i=1}^{n} \frac{1}{n}\delta_i)$. However, we will throughout assume that we are in the limit of a large number of players, so that any individual's choice of strategy has negligible impact on the action distribution $\nu \in \mathbb{P}(Y)$.

Definition 3.2. Let $((X, \mu), Y, \Phi)$ be a game. A *strategy assignment* is a measure $\gamma \in \mathbb{P}(X \times Y)$ with first marginal μ. A strategy assignment γ is called *pure* whenever $\gamma = (id, T)_{\#}\mu$ for some $T : X \to Y$.

Remark. Any strategy assignment γ naturally induces a distribution of actions via its second marginal. We will denote $\nu := (\pi_Y)_{\#}\gamma$ when unambiguous. We interpret γ as encoding that a player is of type in A and plays action in B with probability $\gamma(A, B)$.

Definition 3.3. A strategy assignment γ is a *Cournot–Nash* equilibrium whenever

$$\gamma\left(\{(x, y) \in X \times Y : y \in \operatorname{argmax}_{z \in Y} \Phi(x, z, \nu)\}\right) = 1.$$

Varying a single player x's strategy z while leaving the overall distribution of strategies ν fixed in the $\operatorname{argmax}_{z \in Y} \Phi(x, z, \nu)$ is justified by our remark about there being many players.

Theorem 3.4 (Mas-Colell). *Let $((X, \mu), Y, \Phi)$ be a game. Then there exists a Cournot-Nash equilibrium γ. Furthermore, if Y is finite and $\mu(\{x\}) = 0$ for all $x \in X$, there exists a pure Cournot-Nash equilibrium.*

Proof. The proofs of the first and second claims rely on the Schrauder and Kakutani fixed point theorems, respectively. As such, they are non-constructive. See [MC84] for details. □

[1] Blanchet and Carlier work in somewhat more generality, but for our purposes, Euclidean space suffices.

4 Connection to Optimal Transport

Definition 4.1. We assume that the payoff function $\Phi(x, y, \nu)$ is additively separable as follows:

$$\Phi(x, y, \nu) = b(x, y) + \mathcal{V}[\nu](y).$$

We assume b is continuous on the domain $X \times Y$.

Remark. The map b parametrizes the inherent benefit to a player of type x when taking action y in isolation. On the other hand, $\mathcal{V}[\nu](y)$ is the additional benefit to any individual of taking action y given the population's action distribution ν. $\mathcal{V}$ is called the interaction map.

Lemma 4.2. *Let $((X, \mu), Y, b + \mathcal{V})$ be a game. Let $\gamma \in \mathbb{P}(X \times Y)$, and denote its second marginal by ν. We claim γ is a Cournot-Nash equilibrium if and only if there exists a continuous $\varphi : X \to \mathbb{R}$ satisfying $b(x, y) + \mathcal{V}[\nu](y) \leq \varphi(x)$ for every $(x, y) \in X \times Y$ and equality γ-almost everywhere.*

Proof. One direction is obvious. If we assume γ is a Cournot-Nash equilibrium, then it suffices to take $\varphi(x) := \max_{y \in Y} b(x, y) + \mathcal{V}[\nu](y)$. The converse is less obvious, but [BC16] take it as intuitively clear. □

Proposition 4.3. *Let $((X, \mu), Y, b + \mathcal{V})$ be a game. If γ is a Cournot-Nash equilibrium, then γ solves the optimal transport problem*

$$\gamma \in \text{argmax}_{\hat{\gamma} \in \Gamma(\mu, \nu)} \int_{X \times Y} b(x, y) d\hat{\gamma}(x, y)$$

Proof. Consider the dual problem to the OT problem $\max \int b(x, y)$. It would involve finding Kantorovich potentials φ, ψ. In our setup, we have essentially fixed the second Kantorovich potential to be $\psi = \mathcal{V}[\nu]$, from which we conclude by duality in OT. □

Corollary 4.4. *In a similar spirit to the preceding proposition, we carry over classic results from OT. Let $((X, \mu), Y, b + \mathcal{V})$ be a game. Suppose μ is absolutely continuous and that b satisfies the generalized Spence-Mirrlees condition. Then if there exists a Cournot-Nash equilibrium γ, it is pure and unique.*

Remark. The imposition of the interaction map $\mathcal{V}[\nu]$ as the second Kantorovich potential restricts us from being able to conclude that optimal transport plans (and thus Cournot-Nash equilibria) exist. All we can conclude is that *if* a Cournot-Nash equilibrium exists, it is unique and pure.

5 Extensions

Here we have some extensions proposed by Blanchet and Carlier.

Remark. In many practical applications, there is exclusivity of use for certain strategies. Blanchet and Carlier account for exclusivity of use by introducing a reference measure m_0 on the space of actions Y, and then insisting that we restrict action distributions to those which are absolutely continuous with respect to the reference measure. In [BC16], the authors use the exapmle of choice of holiday destination – not everyone can go to the same beach in Mexico due to overcrowding. Thus not all strategy assignments are admissible. The following definition formalizes this remark.

Definition 5.1. Let $((X, \mu), Y, \Phi)$ be a game, and fix a measure m_0 on Y. Let

$$\mathcal{D} = \{\nu \in \mathbb{P}(Y) : \nu \ll m_0\}.$$

A strategy assignment γ is *admissible* (given m_0) whenever its first marginal is μ, and its second marginal is an element of $\mathcal{D}$.

Remark. In [BC16], the authors note that the requirement that a strategy assignment γ be admissible wreaks havoc on standard proofs of the existence of optimal transport plans and thus Cournot-Nash equilibria. While continuity of the map $\nu \mapsto \mathcal{V}[\nu]$ allows us to apply fixed point theorems (see [MC84]), here our restriction makes the problem of existence more delicate.

Definition 5.2. Assume $\mathcal{D}$ is convex. The map $\nu \mapsto \mathcal{V}[\nu]$ is a *differential* on $\mathcal{D}$ precisely when there exists a map $\mathcal{E} : \mathcal{D} \to \mathbb{R}$ satisfying, for every $(\rho, \nu) \in \mathcal{D}^2$, we have

1. $\mathcal{V}[\nu] \in L^1(\rho)$, and
2. $\lim\limits_{\varepsilon \to 0^+} \frac{\mathcal{E}[(1-\varepsilon)\nu + \varepsilon\rho] - \mathcal{E}[\nu]}{\varepsilon} = \int_Y \mathcal{V}[\nu]\, d(\rho - \nu)$.

In this case, $\mathcal{V}[\nu]$ is called the first variation of $\mathcal{E}$.

Theorem 5.3. *Suppose $\mathcal{V}[\nu]$ is the first variation of some $\mathcal{E} : \mathcal{D} \to \mathbb{R}$. Then under various technical conditions, if ν solves*

$$\sup_{\nu \in \mathcal{D}} \left[\left(\sup_{\gamma \in \Gamma(\mu,\nu)} \int b(x,y) d\gamma(x,y) \right) + \mathcal{E}[\nu] \right]$$

and γ is an optimal coupling of μ and ν, then γ is a Cournot-Nash equilibrium.

Remark. This theorem, proved in [BC16], demonstrates (at a high level) the approach/philosophy employed throughout the entire paper. First, one finds ν by minimization of the expression above. Having determined the optimal strategy distribution ν, one finds the optimal coupling between players and strategies, γ, using optimal transport. Earlier results guarantee that the obtained strategy assignment is a Cournot-Nash equilibrium.

Remark (Computability). A major benefit of the variational approach employed is that it is feasible to numerically compute Cournot-Nash equilibria for sufficiently well behaved $\mathcal{E}$. This is contrasted with general proofs of existence, both in the case of finite games and the case covered by Mas-Colell in [MC84]. All of the aforementioned contexts make use of fixed point theorems, so that existence is guaranteed but we are presented with no way of determining what Cournot-Nash equilibria look like.

Remark (Efficiency vs. stability). From the perspective of economics, two important concepts should be kept in mind: stability and efficiency. Cournot-Nash equilibria are defined as stable points, with no discussion of their efficiency. Blanchet and Carlier give examples to show that, in general, Cournot-Nash equilibria are not efficient; those familiar with the prisoner's dilemma in classical game theory will not find inefficiency in equilibria surprising. Blanchet and Carlier propose a transfer mechanism to ensure efficiency at Cournot-Nash equilibria, and conclude with a nice discussion of the cost of anarchy.

6 Conclusion

In this companion reading to [BC16], we provide the necessary tools/framework for analysing game theoretic concepts in the language of optimal transport. While it is not a priori obvious that game theory and OT might share such a strong connection, this result is in keeping with the surprising breadth of applicability of OT. Blanchet and Carlier argue that "the cross-fertilisation between economics and optimal transport will rapidly develop," citing a recent examples the applications of OT to hedonics and matching problems, multidimensional screening, and urban economics. Furthermore, the fact that Blanchet and Carlier propose methods which are computable lends even more creedence to notion that OT will become more prevalent in applied domains. The research paper by Blanchet and Calier, as well as this companion reading, should help to bridge the gap between pure math, and in particular, OT researchers, and the more applied researchers who might seek to use OT in their economic and game-theoretic pursuits.

References

[BC16] Adrien Blanchet and Guillaume Carlier. Optimal transport and Cournot-Nash equilibria. *Mathematics of Operations Research*, 41(1):125–145, 2016.

[MC84] Andreu Mas-Colell. On a theorem of Schmeidler. *Journal of Mathematical Economics*, 13:201–206, 1984.